Roe algebras and coarse geometry

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Coarse geometry: coarse spaces and coarse maps

In this talk, we will explore the fundamental concepts of coarse geometry and compare them with the framework of geometric group theory introduced by Vincent in the preceding lecture series. We will introduce coarse structures as an abstraction of metric spaces viewed through the lens of large-scale geometry and discuss coarse equivalences, which are maps preserving large-scale properties.

1 Coarse geometry: coarse spaces and coarse maps

The aim of this talk is to introduce large-scale geometry for general metric (coarse) spaces. As Vincent mentioned in the previous series of talks, quasi-isometries provide a suitable notion of equivalence for quasi-length spaces. However, beyond the quasi-length setting, certain complications may arise. To illustrate this, we consider the following examples of non-quasi-length spaces.

The set of natural numbers $\mathbb{N}$ with a disctance $d_k(n,m) = |n^k - m^k|$, for $k > 1$.

One can verify that different choices of k yield non-quasi-isometric spaces. However, when viewed from a large-scale perspective, these spaces appear indistinguishable. Following the fundamental principle of large-scale geometry—namely, that "spaces which appear identical from afar should be regarded as equivalent" – we shall not differentiate between them. Consequently, several notions from geometric group theory must be adapted to accommodate non-quasi-length spaces. To do so, we will introduce the notion of coarse space in the same style as the passage from balls in metric spaces to topology..
Suppose that a metric space (X,d) is given. Consider the collection of all subsets $E \subset X \times X$ for which there exists a constant $R \geq 0$ such that $d(x,y) \leq R$ for every pair $(x,y) \in E$. We denote this collection by $\mathcal{E}_d$. Observe that $\mathcal{E}_d$ satisfies the following properties:

1. The diagonal Δ_X of $X \times X$ belongs to $\mathcal{E}_d$;
2. The set $\mathcal{E}_d$ is stable by finite unions;
3. The set $\mathcal{E}_d$ is stable by inclusions;
4. The set $\mathcal{E}_d$ is stable by compositions of its elements viewed as relations on X (i.e., if $E, F \in \mathcal{E}_d$, then the set $E \circ F = \{(z,x) \mid \exists y \in X \colon (z,y) \in E, (y,x) \in F\}$ belongs to $\mathcal{E}_d$);
5. The set $\mathcal{E}_d$ is stable under transpositions of its elements viewed as relations on X (i.e. if $E \in \mathcal{E}_d$, the set $E^t = \{(x,y) \mid (y,x) \in \mathcal{E}_d\}$ belongs to $\mathcal{E}_d$).

Definition 1.1 (Coarse space). A *coarse space* is a pair $(X, \mathcal{E})$, where X is a set, and $\mathcal{E} \subset \mathcal{P}(X \times X)$ satisfies the five axioms listed above. The elements of $\mathcal{E}$ will be called the controlled sets.

As we have already seen, every metric space can be turned into a coarse space. The coarse structure $\mathcal{E}_d$ introduced above is said to be *a coarse structure induced by metric*. One can proceed analogously with extended metric spaces (i.e. sets equipped with a metric that is allowed to attain infinite values). Hence, we have a significant amount of examples. For instance, let X be a set, consider two coarse structures $\mathcal{E}_{\min} = \langle \Delta_X \rangle$ and $\mathcal{E}_{\max} = \langle X \times X \rangle$. Note that for any coarse structure $\mathcal{E}$ on X one has inclusions

$$\mathcal{E}_{\min} \subset \mathcal{E} \subset \mathcal{E}_{\max}.$$

Hence, we will refer to these coarse structures as the minimal coarse structure and the maximal coarse structure. It is easy to see that the following metrics

$$d_{\min}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \infty, & \text{if } x \neq y, \end{cases} \qquad d_{\max}(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

induce the minimal and the maximal coarse structures, respectively. However not every coarse structure is induced by an extended metric (unless the space is finite). It would be nice to characterise all coarse spaces whose coarse structures are induced by metrics.

Definition 1.2. A coarse space $(X,\mathcal{E})$ is said to be *countably generated* if there are controlled sets $\{E_n\}_{n\in\mathbb{N}}$ such that $\mathcal{E}$ is the minimal coarse structure on X containing $\{E_n\}_{n\in\mathbb{N}}$.

The following lemma solves the preceding question entirely.

Lemma 1.3. A coarse space $(X,\mathcal{E})$ is countably generated if and only if its coarse structure is a metric coarse structure for an extended metric d on X.

Sketch of proof. Note that a metric coarse structure is generated by sets

$$E_n = \{(x,y) \mid d(x,y) \leq n\},$$

as any controlled set is a subset of E_n for some $n \in \mathbb{N}$. Suppose given a countably generated coarse space $(X,\mathcal{E})$ with a generating family $\{E_n\}_{n\in\mathbb{N}}$. Define a new generating family $\{F_n\}_{n\in\mathbb{N}\cup\{0\}}$ as follows:

$$F_0 = \Delta_X; \quad F_n = (F_{n-1} \circ F_{n-1}) \cup E_n \cup E_n^t, \quad n \in \mathbb{N}.$$

The new family is a new generating family as it contains the initial one. Define a map

$$d\colon X \times X \to \mathbb{N}\cup\{0\}, \quad (x,y) \mapsto \inf\{n \in \mathbb{N}\cup\{0\} \mid (x,y) \in F_n\}.$$

It remains to check that d defines a metric (exercise). □

Note that the above lemma provides a metric only up to coarse equivalence, which will be introduced shortly. Indeed, it is easy to construct two different metrics that induce the same coarse structure (take any metric d, a number $k \geq 0$ and consider kd). Another example of nonequivalent metrics that induce the same coarse structure is the set of natural numbers with metrics $d_k(n,m) = |n^k - m^k|$ for $k > 1$. Note that for any $R \geq 0$ the set

$$E_R = \{(n,m) \mid |n^k - m^k| \leq R\}$$

is just a union of the diagonal Δ_X and a finite subset of $X \times X$. Vice-versa, any union of the diagonal Δ_X and a finite subset of $X \times X$ is contained in E_R, for some big enough $R > 0$. Hence for $k > 1$ one has

$$\mathcal{E}_{d_k} = \{E \subset X \times X \mid E \subset \Delta_X \cup K, \text{ for some finite subset } K \text{ of } X \times X\}.$$

Remark 1.4. One can show that a coarse space $(X,\mathcal{E})$ is 1-generated, or equivalently, finitely generated, if and only if there is a graph structure on X for which d is the shortest path metric. In particular, every quasi-length space introduced by Vincent is 1-generated.

There are countable sets with a non-countably generated coarse structure. To build such a space, we need the following notions.

Definition 1.5. Let $(X,\mathcal{E})$ be a coarse space, $E \in \mathcal{E}$ be a controlled set, and $x \in X$ be a point. The E-neighbourhood of x is the set

$$E[x] = \{z \in X \mid (z,x) \in E\}$$

Note that the above definition generalizes the notion of R-neighbourhoods in metric spaces. Precisely, a point $y \in X$ belongs to the R-neighbourhood of $x \in X$ if and only if $d(x,y) \leq R$, which is equivalent to $(y,x) \in E_R$. Hence

$$y \in B_R(x) \iff y \in E_R[x].$$

Though E-neighbourhoods are much more general than R-neighbourhoods in metric spaces. For instance if $E = \{(y,x) \mid d(y,x) = 1\}$, then $E[x] = B_1(x) \setminus \mathrm{Int}(B_1(x))$, so it is a sphere of radious 1 around x.

Definition 1.6. A coarse space $(X, \mathcal{E})$ is said to be *uniformly locally finite* (ulf) if for every controlled set $E \in \mathcal{E}$ there exists a number $N \in \mathbb{N}$ such that for every $x \in X$ one has

$$\max\{|E[x]|, |E^t[x]|\} \leq N.$$

1.7 (Example of non-countably generated coarse structure on $\mathbb{N}$)**.** Consider a set of natural numbers $\mathbb{N}$ with the maximal uniformly locally finite coarse structure $\mathcal{E}_{\text{ulf}}^{\max}$ given by

$$E \in \mathcal{E}_{\text{ulf}}^{\max} \iff \sup_{x \in X} \max\{|E[x]|, |E^t[x]|\} < \infty. \tag{1}$$

To see that this coarse structure is not metric, suppose by contradiction that there is a countable generating set $\{E_n\}_{n \in \mathbb{N}}$. By replacing E_n with $F_1 \circ \ldots \circ F_n$, where

$$F_n = E_n \cup E_n^t \cup \Delta_X,$$

we may assume that $E \subset \mathbb{N} \times \mathbb{N}$ is controlled if and only if $E \subset E_n$ for some $n \in \mathbb{N}$. As $\mathbb{N}$ is infinite, we can pick a sequence $\{x_n\}_{n \in \mathbb{N}}$ the elements of which are pairwise distinct. For every $n \in \mathbb{N}$ choose $z_n \in \mathbb{N}$ such that $(x_n, z_n) \notin E_n$. By (1) the sets $E_n[x_n]$ are finite, therefore we can pick $\{z_n\}_{n \in \mathbb{N}}$ to be a sequence of pairwise disjoint elements of X. Let

$$F = \{(x_n, z_n) \mid n \in \mathbb{N}\}.$$

It trivially satisfies (1) and is not contained in any E_n; hence, we have a contradiction.

We have defined the objects of interest, now it's time to introduce morphisms.

Definition 1.8. Let $(X, \mathcal{E})$, $(Y, \mathcal{F})$ be coarse spaces, $f \colon X \to Y$ be a map.

1. A map f is said to be *coarse*, if for every controlled set $E \in \mathcal{E}$ the set $(f \times f)(E) \in \mathcal{F}$. In metric terms, this can be reformulated as follows. For every $R \geq 0$ there exists $S \geq 0$ such that if $d_X(x, y) \leq R$, then $d_Y(f(x), f(y)) \leq S$;
2. Two maps $f, g \colon X \to Y$ are said to be close (denoted by $f \sim g$) if $(f \times g)(E) \in \mathcal{F}$ for every $E \in \mathcal{E}$. In metric terms this is the same as saying that f and g are at a bounded distance from each other, where the distance is defined by $d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$;
3. The map f is said to be a coarse equivalence if it is coarse and there exists $g \colon Y \to X$ such that $f \circ g \sim \text{id}_Y$ and $g \circ f \sim \text{id}_X$;
4. The map f is said to be a coarse embedding if it is a coarse equivalence on its image equipped with the induced coarse structure (i.e., for a subset $A \subset Y$ declare $E \subset A \times A$ to be controlled if E is controlled as a subset of $Y \times Y$).

It is easy to see that $\sim$ is an equivalence relation on the set of coarse maps (note that it is not an equivalence relation on the set of all maps since, for non-coarse maps, it is not reflexive). We should think of morphisms in the coarse category as $\sim$-equivalence classes of coarse maps. Some authors define the coarse category as $\sim$-equivalence classes of coarse and proper maps

Theorem 1.9. A coarse map $f \colon (X, \mathcal{E}) \to (Y, \mathcal{F})$ is a coarse equivalence if and only if the following two conditions hold

1. (Expansivity): For every $F \in \mathcal{F}$ the set $(f \times f)^{-1}(F)$ belongs to $\mathcal{E}$;
2. (Coboundedness): There exists a controlled set $F_0 \in \mathcal{F}$ such that $F[\text{im}(f)] = Y$.

   Sketch of proof. Suppose given a cobounded, expansive coarse map $f \colon X \to Y$. We can factor it as

   $$X \xrightarrow{f} \text{im}(f) \xrightarrow{i} Y,$$

   where $i \colon \text{im}(f) \to Y$ is an inclusion and $\text{im}(f)$ is equipped with the induced coarse structure from Y. Since f is cobounded, for every $y \in Y$ there exists $x_y \in \text{im}(f)$ such that $(y, x_y) \in F_0$. Let $r \colon Y \to \text{im}(f)$ be a map defined by

   $$r(y) = \begin{cases} y, & \text{if } y \in \text{im}(f); \\ x_y, & \text{otherwise.} \end{cases}$$

Note that since r is a retraction, we have an equality $r \circ i = \mathrm{id}_{\mathrm{im}(f)}$. On the other hand, the following inclusion holds

$$(i \circ r \times \mathrm{id}_Y)(F) \subset F_0^t \circ F \cup F \in \mathcal{F},$$

hence $i \circ r \sim \mathrm{id}_Y$, so i is a coarse equivalence. It remains to show that surjective, expansive, cobounded maps are coarse equivalences. Suppose $f\colon X \to Y$ is such a map. Let $g\colon Y \to X$ be any right inverse of f, i.e. $f \circ g = \mathrm{id}_Y$. Note that $g(y) \in f^{-1}(\{y\})$, hence the following chain of inclusions makes sense:

$$(g \circ f \times \mathrm{id}_X)(E) \subset (g \times f^{-1}) \circ (f \times f)(E) \subset (f \times f)^{-1} \circ (f \times f)(E) \in \mathcal{E}.$$

It follows that $g \circ f \sim \mathrm{id}_X$; hence, f is a coarse equivalence. Vise-versa, given a coarse equivalence $f\colon X \to Y$ with coarse inverse $g\colon Y \to X$ note that since

$$f \circ g \sim \mathrm{id}_Y$$

we obtain the coboundedness condition. Indeed, the set $F_0 = \{(y, f \circ g(y) \mid y \in Y)\}$ is controlled, hence $F_0[\mathrm{im}(f)] = Y$. The expansivity is left as an exercise. □

The preceding theorem suggests that we can define coarse equivalences alike quasi-isometries. The following example of coarse equivalences makes it even more evident.

1.10 (Example)**.** Every quasi-isometry (quasi-isometric embedding) is a coarse equivalence (coarse embedding). Indeed, note that the coboundedness condition in the case of metric spaces can be rewritten as

There exists $N \geq 0$ such that for every $y \in Y$ there exists $x \in X$ such that $d(y, f(x)) \leq N$.

This is precisely the coboundedness condition introduced by Vincent. The condition

There exist constants $A, B > 0$ such that $Ad_X(x, y) - B \leq d_Y(f(x), f(y)) \leq Ad_X(x, y) + B$

tells that $f \times f$ maps pairs (x, y) such that $d(x, y) \leq R$ to pairs $(f(x), f(y))$ satisfying $d(f(x), f(y)) \leq AR + B$, hence f is coarse. Moreover, if $d(f(x), f(y)) \leq R$, then $d(x, y) \leq R/A + B$, hence f is expansive. It is not true that any coarsely equivalent spaces are quasi-isometric. For instance, we have seen that natural numbers $\mathbb{N}$ equipped with metrics

$$d_k(x, y) = |x^k - y^k|, \;\; k > 1$$

are pairwise non-quasi-isometric, but they induce the same coarse structure. Hence, the identity map on $\mathbb{N}$ is a coarse equivalence.

Theorem 1.11 (Vincents notes)**.** Let (X, d), (Y, ∂) be quasi-length metric spaces, then they are quasi-isometric if and only if they are coarsely equivalent. Moreover, for every coarse equivalence $f\colon X \to Y$, there exists a quasi-isometry $g\colon X \to Y$ such that $f \sim g$.